Midterm Exam – Function Spaces
B. Math. III

09 September, 2026

(i) Duration of the exam is 2.5 hours. (ii) The maximum number of points you can score in the exam is 110 (total = 100). (iii) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________

Roll Number: ______________________

1. (20 points) An $N \times N$ matrix C is called *circulant* if each row is obtained from the previous one by a circular shift to the right by one position, i.e. $C_{nk} = c_{n-k \pmod N}$ for some vector $c = (c_0, c_1, \dots, c_{N-1}) \in \mathbb{C}^N$. Let $\omega = e^{2\pi i/N}$ and $v_k = (1, \omega^k, \omega^{2k}, \dots, \omega^{(N-1)k})$ for $k = 0, 1, \dots, N-1$. Prove that each v_k is an eigenvector of C with eigenvalue

$$\lambda_k = \sum_{j=0}^{N-1} c_j\, \omega^{-jk}.$$

Total for Question 1: 20

2. Let $\{r_1, r_2, \dots\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$ and set $I_n = \left[r_n - \frac{1}{4^n},\, r_n + \frac{1}{4^n}\right] \cap [0,1]$. Define $f : I \to \mathbb{R}$ by $f(x) = 1$ if $x \in I_n$ for some n, and $f(x) = 0$ otherwise.
   (a) (15 points) Show that f is an upper function and that $\int_0^1 f(x)\, dx \leq \frac{2}{3}$.
   (b) (10 points) Show that $-f$ is *not* an upper function.

Total for Question 2: 25

3. Let $\{f_n\}$ be a Cauchy sequence in $L^1[0,1]$ with respect to the norm $\|f\|_1 = \int_0^1 |f(x)|\,dx$, and consider a subsequence $\{f_{n_k}\}$ such that $\|f_{n_{k+1}} - f_{n_k}\|_1 \leq \frac{1}{2^k}$ for all $k \geq 1$.
   (a) (10 points) Show that the function
   $$g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|$$
   belongs to $L^1[0,1]$.
   (b) (15 points) Using part (a), show that the series
   $$f_{n_1}(x) + \sum_{k=1}^{\infty} \big(f_{n_{k+1}}(x) - f_{n_k}(x)\big)$$
   converges almost everywhere to a function $f \in L^1[0,1]$, and that $\|f_n - f\|_1 \to 0$.

   Conclude that $L^1[0,1]$ is complete.

Total for Question 3: 25

4. For $t \geq 0$, define
   $$A(t) = \left(\int_0^t e^{-x^2}\,dx\right)^2, \qquad B(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx.$$
   (a) (10 points) Prove that $A(t) + B(t) = \dfrac{\pi}{4}$ for all $t \geq 0$.
   (b) (10 points) Prove that $e^{-x^2} \in L^1([0,\infty))$ with $\displaystyle\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

Total for Question 4: 20

5. (20 points) Let $f \in L^1(\mathbb{R})$ be complex-valued and define its Fourier transform $\hat{f} : \mathbb{R} \to \mathbb{C}$ by $\hat{f}(\xi) = \displaystyle\int_{\mathbb{R}} f(x)\,e^{-2\pi i x\xi}\,dx$. Prove that $\hat{f}$ is continuous on $\mathbb{R}$.

Total for Question 5: 20